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The charges of a twisted brane

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Abstract: The charges of the twisted D-branes of certain WZW models are determined. The twisted D-branes are labelled by twisted representations of the affine algebra, and their charge is simply the ground state multiplicity of the twisted representation. It is shown that the resulting charge group is isomorphic to the charge group of the untwisted branes, as had been anticipated from a K-theory calculation. Our arguments rely on a number of non-trivial Lie theoretic identities.

Keywords: D-branes, Conformal Field Models in String Theory.

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1. Introduction

The dynamics of D-branes in string theory is largely determined in terms of their conserved charges. It is therefore of some significance to determine these charges, as well as the resulting charge groups. For strings propagating on the group manifold G (for which the world-sheet theory is described by a WZW model with symmetry algebra $\mathfrak{g}_k$), the brane charges can be determined directly in terms of the underlying conformal field theory. Indeed, it was argued in [1] that for D-branes that preserve the affine symmetry (and that are then labelled by the integrable highest weight representations $P_+^k(\mathfrak{g})$), the charge q_μ of the brane labelled by μ satisfies

$$\dim(\lambda)\, q_\mu = \sum_{\nu\in P_+^k(\mathfrak{g})} N_{\lambda\mu}{}^{\nu}\, q_\nu \pmod{M}\,. \tag{1.1}$$

Here $\lambda \in P_+^k(\mathfrak{g})$ is an integrable highest weight representation of $\mathfrak{g}_k$, $\dim(\lambda)$ is the Weyl dimension of the corresponding finite dimensional representation of the horizontal subalgebra $\bar{\mathfrak{g}}$, and $N_{\lambda\mu}{}^{\nu}$ are the fusion rules. For finite k this identity is only true modulo some integer M, and the charge group that is carried by these D-branes is then $\mathbb{Z}/M\mathbb{Z}$, where M is the largest positive integer for which (1.1) holds. [Since we can divide the q_μ by a common factor, we are assuming here that the greatest common factor of the q_μ is one.] By taking μ to be the identity representation of $\mathfrak{g}_k$, it follows that

$$q_\lambda = \dim(\lambda)\,. \tag{1.2}$$

The number M is then the largest number for which

$$\dim(\lambda)\dim(\mu) = \sum_{\nu\in P_+^k(\mathfrak{g})} N_{\lambda\mu}{}^{\nu}\,\dim(\nu) \pmod{M} \tag{1.3}$$

holds. This number is always[1] of the form

$$M = \frac{k+h^\vee}{\gcd\{k+h^\vee, L\}}\,, \tag{1.4}$$

where $h^\vee$ is the dual Coxeter number of $\bar{\mathfrak{g}}$. It was shown in [1, 4] that for $\bar{\mathfrak{g}} = A_n$, for which $h^\vee = n+1$, L is given by

$$L_A = \operatorname{lcm}\{1, 2, \ldots, n\}\,. \tag{1.5}$$

The value of M, i.e. L, for the other affine algebras has been determined in [2]

$$\begin{aligned} B_n:&\quad L_B = \operatorname{lcm}\{1,2,\ldots,2n-1\}\\ C_n:&\quad L_C = \operatorname{lcm}\{1,2,\ldots,n,1,3,5,\ldots,2n-1\}\\ D_n:&\quad L_D = \operatorname{lcm}\{1,2,\ldots,2n-3\}\,. \end{aligned} \tag{1.6}$$

The formula for C_n was proven in [2]; the formulae for B_n and D_n (as well as the other algebras) were checked numerically up to very high levels. For completeness we also mention the dual Coxeter numbers for these cases,

$$h^\vee(B_n) = 2n-1\,, \qquad h^\vee(C_n) = n+1\,, \qquad h^\vee(D_n) = 2n-2\,.$$

[1] In [2] it is pointed out there are some exceptional levels, e.g. $k=1$ for so(n), where M is larger than (1.4) would indicate. However, K-theoretic considerations [3] suggest that M is in fact always given by (1.4), even for small k, and that M must also satisfy (1.3) for any *dominant* weight λ.

It has been known for some time that many WZW models also possess D-branes that only preserve the affine symmetry up to some twist. In fact, for each automorphism ω of the corresponding finite dimensional Lie algebra $\bar{\mathfrak{g}}$, there exist ω-twisted D-branes. These D-branes are parametrised by ω-twisted highest weight representations of $\mathfrak{g}_k$ [5, 6, 7]. It was argued in [1] that the charge group of the ω-twisted D-branes of a WZW model is of the form $\mathbb{Z}_{M^\omega}$, where M^ω is characterised by the analogue of (1.1): the ω-twisted D-brane labelled by a has an integer charge q^ω_a, such that

$$\dim(\lambda)\, q^\omega_a = \sum_b \mathcal{N}_{\lambda a}{}^b\, q^\omega_b \pmod{M^\omega}\,, \tag{1.7}$$

where $\mathcal{N}_{\lambda a}{}^b$ are the corresponding NIM-rep coefficients that appear in the Cardy analysis of these branes. M^ω is then the largest integer for which (1.7) holds, assuming again that the greatest common divisor of the q^ω_a is one. Unlike the situation above, none of the labels a plays the role of an identity field, and thus it is not clear *a priori* how to determine q^ω_a, let alone M^ω.

It was argued in [7] (see also [6]) that the NIM-rep coefficients $\mathcal{N}_{\lambda a}{}^b$ can be identified with the twisted fusion rules that describe the fusion of the twisted representation a with the untwisted representation λ to give the twisted representation b. The conformal highest weight spaces of all three representations in question, λ, a and b, form actually representations of the *invariant subalgebra* $\bar{\mathfrak{g}}^\omega$ that consists of the ω-invariant elements of $\bar{\mathfrak{g}}$. (This will be discussed in detail in the following subsection.) As such, the twisted fusion rules $\mathcal{N}_{\lambda a}{}^b$ must be truncations of the tensor product coefficients of $\bar{\mathfrak{g}}^\omega$. The situation is therefore analogous to the situation for the untwisted D-branes, where the fusion rule coefficients are truncations of tensor product coefficients of $\bar{\mathfrak{g}}$. This suggests then that q^ω_a should simply be the dimension of the conformal highest weight space of the ω-twisted highest weight representation a, i.e. the Weyl dimension of the representation of the corresponding invariant subalgebra $\bar{\mathfrak{g}}^\omega$[2]

$$q^\omega_a = \dim_{\bar{\mathfrak{g}}^\omega}(a)\,. \tag{1.8}$$

In this paper we shall show that, up to an appropriate equivalence, this is the unique solution of (1.7), and that it solves (1.7) with $M^\omega = M$. We shall explain this construction in detail for the case of the classical algebras with ω being charge conjugation (or chirality flip for the case of so$(2n)$). Looked at from a purely Lie theoretic perspective, the resulting set of identities is quite remarkable. The generalisation to the two other cases (involving D_4 and E_6) should be straightforward. In the course of the proof we shall also give very simple and explicit formulae for the corresponding NIM-rep graphs, from which it will be manifest that they are truncations of the tensor product coefficients of $\bar{\mathfrak{g}}^\omega$. This lends strong support to the suggestion that they are in fact the twisted fusion rules. Other manifestly integral formulae for these coefficients are given in [9].

[2] A related proposal was put forward in [8], based on an analysis for large level. However, the proposal of [8] differs from ours in that for su$(2n+1)$, they take the relevant 'invariant subalgebra' to be so$(2n+1)$, whereas it is sp$(2n)$ in our case.

It is believed that the D-brane charge groups coincide with certain untwisted and twisted geometric K-theory groups [10]–[13]. The latter have been recently calculated [14, 3], and it was found that $M^\omega = M$. Our conformal field theory calculation is therefore in nice agreement with this result.

The paper is organised as follows. In the remainder of this section we collect some general observations about the structure of the twisted algebras, motivate our ansatz, and summarise our results. In section 2, the case of su(3) is analysed in detail. This is generalised to su$(2n+1)$ in section 3. Section 4 deals with the analysis for su$(2n)$, and the case of so$(2n)$ is described in section 5. In section 6 we show that these solutions are in fact unique, and section 7 contains our conclusions. There are a number of appendices in which the details of some of the more Lie theoretic calculations have been collected.

1.1 Background information

We assume the reader is familiar with the basics of non-twisted affine algebras (for this see e.g. [15, 16]). Let $\mathfrak{g}_k$ be any affine (non-twisted) algebra, with colabels $a_i^\vee$, $i = 0, \ldots, n$. Then the dual Coxeter number $h^\vee$ equals $\sum_{i=0}^n a_i^\vee$, and the level k weights $\lambda \in P_+^k(\mathfrak{g})$ are all $(\lambda_0; \lambda_1, \ldots, \lambda_n) = \sum_{i=0}^n \lambda_i \Lambda_i$, which are dominant (i.e. have $\lambda_i \geq 0$, $\lambda_i \in \mathbb{Z}$), and obey $\sum_{i=0}^n \lambda_i a_i^\vee = k$. We will usually identify the affine weight $(\lambda_0; \lambda_1, \ldots, \lambda_n)$ with the finite weight $[\lambda_1, \ldots, \lambda_n]$.

Suppose now that ω is an outer automorphism of the underlying finite dimensional Lie algebra $\bar{\mathfrak{g}}$. A representation a of the twisted algebra $\mathfrak{g}^\omega$ is the same as a twisted representation of the algebra $\mathfrak{g}$ (see for example [17] for an introduction into these matters). If we choose a basis for the finite dimensional Lie algebra $\bar{\mathfrak{g}}$ to consist of eigenvectors of the automorphism ω, then this is a representation for which the modes J_n^a for which a has ω-eigenvalue $+1$ have $n \in \mathbb{Z}$, while the modes J_r^b for which b has ω-eigenvalue -1 have $r \in \mathbb{Z} + \frac{1}{2}$. [We are assuming here for simplicity that ω is of order two, as will be the case for the examples discussed in this paper.] Let us denote by $\bar{\mathfrak{g}}^\omega$ the *invariant subalgebra* of $\bar{\mathfrak{g}}$, i.e. the subalgebra that consists of the ω-invariant generators. The generators in $\bar{\mathfrak{g}}^\omega$ commute with L_0, and they act on the space of conformal highest weight states of a. In fact, if a is an irreducible representation of $\mathfrak{g}^\omega$, the conformal highest weight states will form an irreducible representation of $\bar{\mathfrak{g}}^\omega$, and the representation a is uniquely characterised in terms of this representation. Thus we can think of the labels that describe the integrable ω-twisted representations of $\mathfrak{g}_k$ as describing representations of $\bar{\mathfrak{g}}^\omega$.

1.1.1 Untwisted and twisted D-branes

In this paper we shall consider strings propagating on the simply-connected group G, for which the corresponding conformal field theory is just the charge conjugation modular invariant associated to $\mathfrak{g}_k$. In this case, the untwisted D-branes are labelled by the highest weight representations in $P_+^k(\mathfrak{g})$, and are simply given by the Cardy formula

$$||\lambda\rangle\rangle = \sum_{\mu \in P_+^k} \frac{S_{\lambda\mu}}{\sqrt{S_{0\mu}}} \, |\mu\rangle\rangle \,, \tag{1.9}$$

where $|\mu\rangle\rangle$ is the Ishibashi state in the representation $\mathcal{H}_\mu \otimes \bar{\mathcal{H}}^*_\mu$, and $S_{\lambda\mu}$ is the modular S-matrix. (For an introduction into these matters see for example [18].) The open string that stretches between the branes labelled by λ and μ then contains the representation $\mathcal{H}_\nu$ of $\mathfrak{g}_k$ with multiplicity

$$N_{\nu\mu}{}^{\lambda} = \sum_{\rho\in P^k_+} \frac{S_{\mu\rho}\, S_{\nu\rho}\, S^*_{\lambda\rho}}{S_{0\rho}} . \tag{1.10}$$

Because of the Verlinde formula, these numbers are just the fusion rules of $\mathfrak{g}_k$.

Suppose now that ω is an outer automorphism of the finite dimensional Lie algebra $\bar{\mathfrak{g}}$. Then ω-twisted Ishibashi states exist for the subset $\mathcal{E}$ of *exponents*: this consists simply of the ω-invariant representations of P^k_+. It was argued in [7] (see also [5, 6, 19]) that the ω-twisted D-branes are then labelled by the integrable ω-twisted representations a of $\mathfrak{g}_k$, and that they are explicitly given as

$$||a\rangle\rangle^\omega = \sum_{\mu\in\mathcal{E}} \frac{\psi_{a\mu}}{\sqrt{S_{0\mu}}}\, |\mu\rangle\rangle^\omega , \tag{1.11}$$

where ψ describes the modular transformation of twisted and twining characters. In order to describe this in more detail, we define, for each $\mu \in \mathcal{E}$, the *twining character*

$$\chi^{(\omega)}_\mu(\tau) = \mathrm{Tr}_{\mathcal{H}_\mu}\left(\tau_\omega q^{L_0-\frac{c}{24}}\right), \tag{1.12}$$

where τ_ω is the induced action of ω on $\mathcal{H}_\mu$, the representation space corresponding to μ. An important and nontrivial fact is that the twining character agrees precisely with the ordinary character of the so-called orbit Lie algebra $\breve{\mathfrak{g}}$ [20]. Upon a modular S-transformation we then have

$$\chi^{(\omega)}_\mu\left(-\frac{1}{\tau}\right) = \sum_a \psi_{a\mu}\, \chi_a(\tau) , \tag{1.13}$$

where the χ_a are characters of the twisted algebra $\mathfrak{g}^\omega$, and the modular matrix $\psi_{a\mu}$ is the matrix that appears in (1.11). The open string that stretches between the branes labelled by b and a then contains the representation $\mathcal{H}_\nu$ of $\mathfrak{g}_k$ with multiplicity

$$\mathcal{N}_{\nu a}{}^{b} = \sum_{\rho\in\mathcal{E}} \frac{\psi_{a\rho}\, S_{\nu\rho}\, \psi^*_{b\rho}}{S_{0\rho}} . \tag{1.14}$$

The consistency of the construction requires that these numbers are non-negative integers, and in fact, they must define a *NIM-rep* of the fusion algebra — see e.g. [18, 21] for an introduction to these matters. The positivity of these numbers at least for A_n is a consequence of the subfactor realisation of this NIM-rep [22]. It was argued in [7] (see also [6]) that these numbers agree in fact with the twisted fusion rules. (The fusion of twisted representations is for example discussed in [23].) At any rate, these are the numbers that appear in the definition of the twisted charges (1.7).

Explicit formulae for the ψ-matrix, as well as for the NIM-reps $\mathcal{N}_{\lambda a}{}^{b}$, were given in [7]. Actually, a dramatic simplification occurs if we restrict in the NIM-rep formulae λ to the fundamental weights $\lambda = \Lambda_i$ (see also [19]). This does not lose any information since the character ring of $\bar{\mathfrak{g}}$ is generated by the characters of the Λ_i, and we can therefore write *any* NIM-rep coefficient $\mathcal{N}_{\lambda a}{}^{b}$ in terms of the various coefficients $\mathcal{N}_{\Lambda_i a'}{}^{b'}$. More precisely, write the $\bar{\mathfrak{g}}$-character ch_λ as a polynomial $P_\lambda(\mathrm{ch}_{\Lambda_1}, \mathrm{ch}_{\Lambda_2}, \ldots)$ in the fundamentals; then the NIM-rep matrix $\mathcal{N}_\lambda$ equals $P_\lambda(\mathcal{N}_{\Lambda_1}, \mathcal{N}_{\Lambda_2}, \ldots)$. For all of the NIM-reps considered in this paper, we will explicitly write simple expressions for the $\mathcal{N}_{\Lambda_i a}{}^{b}$.

As an aside, note that the limit of (1.13) as $\tau \to i\infty$ (i.e. $q \to 0$) will be dominated by the contribution from the unique weight of $\mathfrak{g}^\omega$, namely $a = 0$, with minimal conformal weight. The dominant term on the right hand side of (1.13) in this limit is $\psi_{0\mu} q^{-c/24}$. If we take the τ limit along the imaginary axis, the left-side is manifestly positive (being an ordinary character of the orbit Lie algebra). Thus we obtain that $\psi_{0\mu} \geq 0$ for all μ. Similarly, $\psi_{a0} \geq 0$ for all a. Both of these are consistent with the observations of [7]; the positivity of ψ_{a0} is also required by its interpretation in [24] as a boundary entropy.

1.1.2 More details about the relevant twisted algebras

In this paper we shall only deal with the classical algebras that have a non-trivial order-2 automorphism (e.g. charge-conjugation). The relevant algebras $\bar{\mathfrak{g}}$ are then su(n), and so$(2n)$. The invariant subalgebra $\bar{\mathfrak{g}}^\omega$ of su$(2n+1)$ and su$(2n)$ is C_n, while the invariant subalgebra of so$(2n)$ is B_n. This can be read off from the structure of the weights of the twisted algebra. For example, the data on page 81 of [15] can be used to compute e.g. the inner products $(\Lambda_i|\Lambda_j)$ of fundamental weights (these are defined to be dual to the coroots) and hence identify $a = \sum a_i \Lambda_i$ *etc.*

Similarly, one can also identify the labels of the ω-invariant weights μ with weights of some finite dimensional subalgebra. For the case of su$(2n+1)$, μ can be identified with a weight of C_n, while for su$(2n)$, μ is a weight of B_n, and for so$(2n)$, μ is a weight of C_n. [Another way to make these identifications is to compute the conformal weights of the exponents μ, i.e. to compute the norms $(\mu+\rho|\mu+\rho)$ of the weights $C\mu = \mu$, and to compare them with the conformal weights in C_n and B_n, respectively.]

Thus the labels μ and a appear symmetrically for su$(2n+1)$, while they are genuinely different for su$(2n)$ and so$(2n)$. This can be traced back to the fact that both the rows and columns of the ψ matrix for su$(2n+1)$ label representations of $A_{2n}^{(2)}$. On the other hand, for su$(2n)$ the rows label representations of $A_{2n-1}^{(2)}$ while the columns label representations of $D_{n+1}^{(2)}$, and for so$(2n)$ the rows label representations of $D_n^{(2)}$ while the columns label representations of $A_{2n-3}^{(2)}$.

Finally, it remains to determine the levels at which these invariant subalgebras appear. The easiest way to anticipate the relevant shift in level, is to note that all the relevant formulae (e.g. the modular matrices) depend directly on $\kappa = k + h^\vee$ rather than k itself. Thus if we convert say su$(2n+1)$ data (where $\kappa = k + 2n + 1$) into so$(2n+1)$ data (where $\kappa = k_B + 2n - 1$), we should shift the effective level to $k_B = k + 2$.

1.2 Summary of our results

Let us state clearly our results and assumptions. There are three conjectures that are required for some parts of our arguments. These are:

Conjecture B: the largest number m (call it m_B) such that

$$\dim(\lambda)\dim(\mu) = \sum_{\nu} N_{\lambda\mu}^{\nu} \dim(\nu) \pmod{m} \tag{1.15}$$

for all B_n dominant weights λ and all $\mu \in P_+^k(B_n)$, is $M_B = (k+2n-1)/\gcd\{k+2n-1, L_B\}$ for L_B given in (1.6).

Conjecture D: the largest number m (call it m_D) such that (1.15) holds for all D_n dominant weights λ and all $\mu \in P_+^k(D_n)$, is $M_D = (k+2n-2)/\gcd\{k+2n-2, L_D\}$ for L_D given in (1.6).

Conjecture $\mathbf{B}^{\text{spin}}$: the largest number m (call it m_B^{spin}) such that (1.15) holds for all B_n non-spinor dominant weights λ and all spinors $\mu \in P_+^k(B_n)$ (i.e. μ_n is odd and λ_n is even), is $2^n M_B$.

Conjectures B and D were made in [2] and are (to our knowledge) still unproven, although the analogues for A_n and C_n have been proven. There is considerable numerical support for their general validity, and the fact that they fit so nicely into our context lends further support. Conjecture B^{spin} is a natural generalisation of Conjecture B. In appendix D we prove it (i.e. reduce it to Conjecture B) whenever 4 does not divide M_B. We also show m_B^{spin} must equal some power of 2 times M_B, and must divide $2^n M_B$. In [2] a generating set for each fusion ideal $\mathcal{I}_k$ is conjectured; that conjecture implies

$$m_B^{\text{spin}} = \gcd\{\dim_B((k-2)\Lambda_1 + \Lambda_i + \Lambda_n)\}_{1<i<n}\,. \tag{1.16}$$

Our assertion about the twisted charge group consists of two separate statements:

(i) eq. (1.8) satisfies (1.7) with $M^\omega = M$;

(ii) eq. (1.8) is the unique solution to (1.7), up to rescaling. (Of course, given any solution q_a^ω to (1.7), another will be given by $q_a^{\omega\prime} := c\, q_a^\omega$ for any number c. What (ii) says is that any solution to (1.7) is of that form, for q_a^ω given by (1.8).)

We should note that M^ω always has to be a factor of M.[3] By construction, M is the greatest common divisor of the dimensions of elements in the fusion ideal, the ideal by which we have to quotient the representation ring in order to obtain the fusion ring. Since the NIM-rep is a representation of the fusion ring, any element of the fusion ideal must act trivially. Thus for any α in the fusion ideal and for any twisted D-brane a, $\dim(\alpha)\, q_a^\omega = 0 \,(\mathrm{mod}\ M^\omega)$. Together with the fact that the q_a^ω must be coprime, this implies that M^ω must be a factor of M.

[3]The following argument is due to Stefan Fredenhagen; we are grateful to him for communicating this argument to us.

For $\bar{\mathfrak{g}} = \mathrm{su}(2n+1)$, when the level k is odd, we prove (i) and (ii), with no additional assumptions.

For $\bar{\mathfrak{g}} = \mathrm{su}(2n+1)$, when the level k is even, we require Conjecture B. More precisely, we only need that conjecture when we state that the twisted charge group M^ω, which we show equals m_B, in fact equals the untwisted M_A.

For $\bar{\mathfrak{g}} = \mathrm{su}(2n)$, when the level is odd, we again only require Conjecture B. For $\bar{\mathfrak{g}} = \mathrm{su}(2n)$, when the level is even, we require in addition Conjecture $\mathrm{B}^{\mathrm{spin}}$ for the proof of (i).

For $\bar{\mathfrak{g}} = \mathrm{so}(2n)$, (i) follows from Conjectures B and D only, unless 2^{n+1} divides M_B, in which case Conjecture $\mathrm{B}^{\mathrm{spin}}$ is also needed. Only Conjectures B and D are needed for (ii).

2. The construction for su(3)

Let us first illustrate the construction in the simplest non-trivial case, where $\bar{\mathfrak{g}} = \mathrm{su}(3)$. As follows from (1.4), the untwisted charge lattice is $\mathbb{Z}/M\mathbb{Z}$, where $M = k+3$ if k is even, and $M = (k+3)/2$ if k is odd. The NIM-rep matrices $\mathcal{N}_{\Lambda_i a}{}^b$ involving both fundamental weights $\Lambda_1 = \mathbf{3}$ and $\Lambda_2 = \bar{\mathbf{3}}$ are described by the 'twisted fusion graph' of figure 1 [25, 7]. (From now on we shall call this NIM-rep the twisted fusion rules; our analysis does, however, not rely on the assumption that this identification is in fact correct.)

Let us first consider the case when k is even. There are $k/2+1$ twisted representations (corresponding to each of the nodes on the fusion graph), which we may label left to right by $a = 0, 1, \ldots, k/2$. The twisted fusion rules are uniquely determined in terms of the fusion with the fundamental representation $\mathbf{3}$ of su(3), and are given as

$$\begin{aligned}
\mathbf{3} \otimes [0] &= [0] + [1]\,, \\
\mathbf{3} \otimes [n] &= [n-1] + [n] + [n+1]\,, \qquad n = 1, \ldots, \frac{k}{2} - 1\,, \\
\mathbf{3} \otimes \left[\frac{k}{2}\right] &= \left[\frac{k}{2} - 1\right].
\end{aligned} \tag{2.1}$$

These fusion rules imply that the corresponding charges $q_a \equiv q_a^\omega$, where ω is charge conjugation, must satisfy

$$\begin{aligned}
3\, q_0 &= q_0 + q_1 \\
3\, q_n &= q_{n-1} + q_n + q_{n+1} \qquad n = 1, \ldots, \frac{k}{2} - 1\,, \\
3\, q_{k/2} &= q_{k/2-1}\,,
\end{aligned} \tag{2.2}$$

where all equalities are understood to be modulo some as-yet-undetermined number M^ω. Starting with the initial condition $q_0 = 1$ say, the unique solution (mod M^ω) to these recursion relations is

$$q_n = n + 1\,. \tag{2.3}$$

In the next section we will interpret this as a dimension. These values satisfy all but the last equation identically; the last equation then gives

$$3\left(\frac{k}{2} + 1\right) = \frac{k}{2} \quad (\mathrm{mod}\ M^\omega)\,, \tag{2.4}$$

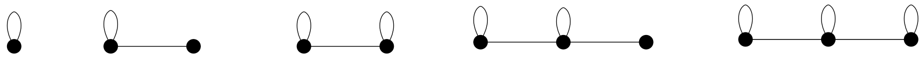

Figure 1: The NIM-rep graphs for su(3) with charge conjugation for $k = 1, 2, 3, 4, 5$.

and thus we should take $M^\omega = k + 3$, in agreement with the value (1.4) for M. If we had instead performed the analysis starting with the 'initial condition' $q_{k/2} = 1$ (which would have been natural had we numbered the nodes of figure 1 from right to left), we would have obtained the unique answer

$$\hat{q}_n = 2\left(\frac{k}{2} - n\right) + 1\,, \tag{2.5}$$

again with $M^\omega = k + 3$. This solution is equivalent to (2.3) since

$$\hat{q}_n = (k + 3) - 2(n + 1) = M^\omega - 2q_n\,. \tag{2.6}$$

Indeed, since the charges q_n are only defined modulo M^ω, the two solutions differ by multiplication by $l = -2$. Since $l = -2$ is coprime to $M^\omega = k + 3$ here, the two solutions are mathematically interchangeable. More generally, we call two sets of charges q^ω_a and $\hat{q}^\omega_a$ *equivalent* whenever

$$\hat{q}^\omega_a = l\, q^\omega_a \pmod{M^\omega}\,, \tag{2.7}$$

for some $l \in \mathbb{Z}$ coprime to M^ω. This is the obvious redundancy in the solutions to (2.2). Nevertheless, 'equivalent' solutions are not equally satisfactory: (2.3) has a direct interpretation in terms of dimensions which is much more obscure in (2.5). Starting from the fusion graphs of figure 1, it is unclear which of these mathematically equivalent solutions is physically preferred. However, from the analysis of [7] it is clear that the left-most vertex (what we called $a = 0$) corresponds to the identity and should be assigned charge $q_0 = 1$. We will see in the next section that the first equation $\mathbf{3} \otimes [0] = [0] + [1]$ then corresponds to the $su(2) \subset su(3)$ branching rule $\mathbf{3} = \mathbf{1} + \mathbf{2}$.

The analysis for k odd is essentially the same. For k odd, there are $(k + 1)/2$ twisted representations that we label by the integers $a_1 = 0, 1, \ldots, (k - 1)/2$. The twisted fusions with $\mathbf{3}$ are the same as for even k, except that now the last fusion is

$$\mathbf{3} \otimes \left[\frac{k-1}{2}\right] = \left[\frac{k-3}{2}\right] + \left[\frac{k-1}{2}\right]. \tag{2.8}$$

The recursion formula still implies that the charges are given by (2.3), but now the last equation gives

$$(k + 1) = \frac{k-1}{2} \pmod{M^\omega}\,, \tag{2.9}$$

which therefore implies that the maximal choice for M^ω is $M^\omega = (k+3)/2$, and thus agrees with the untwisted M.

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3. The analysis for $\mathrm{su}(2n+1)$

Let us now generalise the previous discussion to $\mathrm{su}(2n+1) = A_{2n}$ with $\omega = C$ being charge conjugation. As was explained in [5, 7], the ω-twisted boundary states are labelled by the level k ω-twisted weights of $\mathfrak{g} = \widehat{\mathrm{su}}(2n+1)$, or alternatively the level k weights of the twisted Lie algebra $\mathfrak{g}^\omega = A^{(2)}_{2n}$. They can be equated with all $(n+1)$-tuples $(a_0; a_1, \ldots, a_n)$ where $k = a_0 + 2a_1 + 2a_2 + \cdots + 2a_n$ and $a_i \in \mathbb{Z}_{\geq 0}$. The ground states of this twisted representation (i.e. the states of lowest conformal weight) form an irreducible representation of the invariant subalgebra $\bar{\mathfrak{g}}^\omega = C_n$ with highest weight $[a_1, \ldots, a_n]$. We now propose that the corresponding D-brane charge is simply the Weyl dimension of this irreducible representation, i.e.

$$q^\omega_a = \dim_C([a_1, \ldots, a_n]) = \dim_C(a) \,. \tag{3.1}$$

In this section we shall prove, assuming conjecture B for k even, that this solves (1.7) with $M^\omega = M$. In section 6 we address uniqueness. For $n = 1$, the invariant subalgebra is $C_1 = \mathrm{su}(2)$, and the right hand side is just $a_1 + 1$, in agreement with (2.3) of the previous section.

Our proof that (3.1) solves (1.7) depends on whether k is even or odd, and we will deal with these cases in turn.

3.1 The case of odd level

When the level k is odd, a simplification occurs in that we can identify (bijectively) each boundary label a with the $\widehat{C}_n$ level $k - 1/2$ weight $\widetilde{a} = (a_0 - 1/2; a_1, \ldots, a_n) \equiv [a_1, \ldots, a_n]$, and similarly for μ.[4] (For convenience, we shall use tildes to denote quantities associated with $\widehat{C}_n$ level $k - 1/2$.) With this, the NIM-rep becomes [7]

$$\mathcal{N}_{\lambda a}{}^{b} = \sum_{\tilde{\gamma}} b^{\lambda}{}_{\tilde{\gamma}} \, \widetilde{N}_{\tilde{\gamma}\tilde{a}}{}^{\tilde{b}} \,, \tag{3.2}$$

where $\oplus\, b^{\lambda}{}_{\tilde{\gamma}}\,(\tilde{\gamma}) = (\lambda)$ are the branching rules for $C_n \subset A_{2n}$. The fusion coefficients $\widetilde{N}_{\tilde{\gamma}\tilde{a}}{}^{\tilde{b}}$ are defined in the obvious way for $\tilde{\gamma}$ not necessarily in $\tilde{P}_+ = P_+^{(k-1)/2}(C_n)$ (using e.g. the fact that $S_{\lambda\mu}$ is affine Weyl antisymmetric in λ and μ), in which case they can be negative. For the fundamental representations (Λ_i) of A_{2n}, the branching rules are particularly simple:

$$(\Lambda_i) = (\Lambda_{2n+1-i}) = [0, \ldots, 0] + \bigoplus_{j=1}^{i} \tilde{\Lambda}_j \qquad i \leq n \,, \tag{3.3}$$

where $\tilde{\Lambda}_j$ is the jth fundamental representation of C_n. (Of course, the left hand side of (3.3) refers to the restriction of the A_{2n} representations to the C_n subalgebra.) As a consequence of (3.3), the corresponding NIM-reps are simply

$$\mathcal{N}_{\Lambda_i a}{}^{b} = \mathcal{N}_{\Lambda_{2n+1-i} a}{}^{b} = \delta_a{}^b + \sum_{j=1}^{i} \tilde{N}_{\tilde{\Lambda}_j \tilde{a}}{}^{\tilde{b}} \qquad i \leq n \,. \tag{3.4}$$

This formula was already found in [19]. For completeness we mention that the NIM-rep coefficients for the simple currents J^i are all trivial: $\mathcal{N}_{J^i}{}_a{}^b = \delta_a{}^b$.

[4] The integrable highest weights of $\widehat{C}_n$ level $\tilde{k}$ are the weights $\tilde{\mu} = (\tilde{\mu}_0; \tilde{\mu}_1, \ldots, \tilde{\mu}_n)$ for which $\sum_{i=0}^{n} \tilde{\mu}_i = \tilde{k}$.

As an aside, we note that it is manifest from (3.2) that the NIM-rep coefficient ${\mathcal{N}_{\lambda a}}^b$ equals (for sufficiently large k) the tensor product coefficient ${T_{\lambda a}}^b$ for the invariant subalgebra $\bar{\mathfrak{g}}^\omega = C_n$. This lends strong support to the assertion that these NIM-rep coefficients are in fact the twisted fusion rules.

With the ansatz (3.1) for q_a^ω, we can now rewrite the right hand side of (1.7) as

$$\begin{aligned}\sum_b {\mathcal{N}_{\lambda a}}^b q_b^\omega &= \sum_{\tilde{\gamma}} {b^\lambda}_{\tilde{\gamma}} \sum_{\tilde{b}} {\tilde{N}_{\tilde{\gamma}\tilde{a}}}^{\tilde{b}} \dim_C(\tilde{b}) \\ &= \sum_{\tilde{\gamma}} {b^\lambda}_{\tilde{\gamma}} \dim_C(\tilde{\gamma}) \dim_C(\tilde{a}) \qquad (\mathrm{mod}\ M_C) \\ &= \dim_A(\lambda) \dim_C(\tilde{a}) \quad (\mathrm{mod}\ M_C)\,, \end{aligned} \tag{3.5}$$

where M_C is the untwisted charge number corresponding to C_n at level $(k-1)/2$ (see (1.6)), which was proved in [2] to be

$$M_C = \frac{(k-1)/2+n+1}{\gcd\{(k-1)/2+n+1, L_C\}}\,, \tag{3.6}$$

where $L_C = \mathrm{lcm}\{1,2,\ldots,n,1,3,\ldots,2n-1\} = \mathrm{lcm}\{1,2,\ldots,2n\}/2$. Thus the denominator (like the numerator) of (3.6) is half that of (1.4) for A_{2n}, and it follows that M_C agrees with the untwisted M for A_{2n} at odd level k.

3.2 The case of even level

If k is even, the previous analysis is not available, but one can still regard the ψ-matrix as a symmetric submatrix of the S-matrix for, in this case, $\widehat{B}_n$ level $k+2$. More specifically, identify the boundary state a with $a' = (a_0+a_1+1; a_1, \ldots, a_{n-1}, 2a_n+1) \in P'_+ \equiv P_+^{k+2}(B_n)$.[5] (For convenience we shall use primes to denote quantities associated with $\widehat{B}_n$ level $k+2$.) Using this identification, we can express [7] the NIM-rep coefficients ${\mathcal{N}_{\lambda a}}^b$ in terms of the ordinary fusion coefficients N' of $\widehat{B}_n$ level $k+2$, and branching rules for $B_n \subset A_{2n}$, namely

$${\mathcal{N}_{\lambda a}}^b = \sum_{\gamma'} {b^\lambda}_{\gamma'} \left({N'_{\gamma' a'}}^{b'} - {N'_{\gamma' a'}}^{J' b'} \right). \tag{3.7}$$

Here, J' is the simple current of $\widehat{B}_n$, which acts on weights ν' by $J'\nu' = (\nu'_1; \nu'_0, \nu'_2, \ldots, \nu'_n)$. The coefficients ${b^\lambda}_{\gamma'}$ describe the branching rules $\oplus_{\gamma'} b^\lambda_{\gamma'}(\gamma') = (\lambda)$ for the embedding of $B_n \subset A_{2n}$. Note that these γ' necessarily will be non-spinors. The sum in (3.7) is over all dominant weights γ' of B_n; as before we extend the definition of the fusion coefficients ${N'_{\gamma' a'}}^{b'}$ to arbitrary dominant weights γ' in the obvious way. Again, for the fundamental representations of A_{2n}, the branching rules are very simple

$$\begin{aligned} (\Lambda_i) &= (\Lambda_{2n+1-i}) = (\Lambda'_i) \qquad i<n \\ (\Lambda_n) &= (\Lambda_{n+1}) = (2\Lambda'_n)\,. \end{aligned} \tag{3.8}$$

Thus, as before, (3.7) gives a very simple and explicit formula for the corresponding NIM-reps:

$${\mathcal{N}_{\Lambda_i a}}^b = {\mathcal{N}_{\Lambda_{2n+1-i} a}}^b = {N'_{\Lambda'_i a'}}^{b'}\,, \tag{3.9}$$

[5] $P_+^{k'}(B_n)$ consists of the weights $\mu' = (\mu'_0; \mu'_1, \ldots, \mu'_n)$ for which $\mu'_0 + \mu'_1 + 2\sum_{i=2}^{n-1} \mu'_i + \mu'_n = k'$.

with the obvious analogue for $\mathcal{N}_{\Lambda_n}$. These formulae were already found in [19]. For completeness we also mention that the NIM-rep coefficients of the simple currents are trivial: $\mathcal{N}_{J^i\ a}{}^b = \delta_a{}^b$.

For su(3), the above requires some clarification. By 'B_1 level $k+2$' here we mean su(2) at level $2(k+2)$; $(a_0;a_1)' = (2a_0+2a_1+3;2a_1+1)$ and $(\mu_0;\mu_1)' = (2\mu_0+2\mu_1+3;2\mu_1+1)$. The simple current acts as $J'(\nu_0';\nu_1') = (\nu_1';\nu_0')$; (3.8) becomes $(\Lambda_1) = (\Lambda_2) = (2\Lambda_1')$.

As an aside we note that the NIM-rep coefficient $\mathcal{N}_{\lambda a}{}^b$ in (3.7) does indeed equal the tensor product coefficient $T_{\lambda a}{}^b$ for the invariant subalgebra $\bar{\mathfrak{g}}^\omega = C_n$, for sufficiently large level, as should be the case if the NIM-rep agrees with the twisted fusion rules; this is illustrated for the case of su(7) in appendix C. The proof of this not-completely-obvious fact follows the analogous argument sketched in the next section for su$(2n)$. Incidentally, the large k limit of (3.7) tells us that the C_n tensor product coefficient $T_{\lambda a}{}^b$ equals the B_n tensor product coefficient $T'_{\lambda a'}{}^{b'}$ — something which is not *a priori* obvious.

Let us first make the ansatz (as we shall see later on this will turn out to be equivalent to (3.1))

$$\hat{q}^\omega_a = \dim_B\left([a_1,\ldots,a_{n-1},2a_n+1]\right) = \dim_B(a')\,, \tag{3.10}$$

where the right hand side is the B_n dimension of a'. Let $\mathcal{G}$ be the subset of $P_+^{k+2}(B_n)$ consisting of the images b' of boundary states b under $b\mapsto b'$. More explicitly,

$$\mathcal{G} \equiv \{(b_0;b_1,\ldots,2b_n+1)\in P'_+\ :\ 2\sum_{i=1}^n b_i \le k\}\,. \tag{3.11}$$

Then the right hand side of (1.7) becomes

$$\begin{aligned}\sum_b \mathcal{N}_{\lambda a}{}^b\,\hat{q}^\omega_b &= \sum_{\gamma'} b^\lambda{}_{\gamma'} \sum_{b'\in\mathcal{G}}\left[N'_{\gamma' a'}{}^{b'}\,\dim_B(b') - N'_{\gamma' a'}{}^{J'b'}\,\dim_B(b')\right]\\ &= \sum_{\gamma'} b^\lambda{}_{\gamma'}\left[\sum_{b'\in\mathcal{G}} N'_{\gamma' a'}{}^{b'}\,\dim_B(b') - \sum_{J'b'\in\mathcal{G}} N'_{\gamma' a'}{}^{b'}\,\dim_B(J'b')\right]\\ &= \sum_{\gamma'} b^\lambda{}_{\gamma'}\left[\sum_{b'\in\mathcal{G}} N'_{\gamma' a'}{}^{b'}\,\dim_B(b') + \sum_{J'b'\in\mathcal{G}} N'_{\gamma' a'}{}^{b'}\,\dim_B(b')\right] \pmod{M_B}\,,\end{aligned} \tag{3.12}$$

where we have used in the final line that $\dim(J'b') = -\dim(b') \pmod{M_B}$, as is shown in appendix B. If $b'\in\mathcal{G}$, then $\hat{b} = J'b'\notin\mathcal{G}$, since $\hat{b}_1 = k+1-b_1-2\sum_{i=2}^n b_i$, and thus

$$2\sum_{i=1}^n \hat{b}_i = 2k+2-2\sum_{i=1}^n b_i \ge k+2\,. \tag{3.13}$$

Conversely, if $b'\notin\mathcal{G}$, then either $J'b'\in\mathcal{G}$, or $J'b' = b'$ (i.e. $b_0' = b_1'$). Thus P'_+ consists of the three disjoint sets $\mathcal{G}$, $J'\mathcal{G}$, and the J'-fixed points $f = (f_1;f_1,f_2,\ldots,f_n)$. Given that a' is a spinor weight while γ' is not, b' must be a spinor weight in order for the fusion rule coefficient to be non-trivial. Since k is even, there are no spinor weights that are fixed points of J' (in particular, $f_n = k \pmod 2$ for fixed points of $\widehat{B}_n$ at level $k+2$). Thus we can replace (3.12) with

$$
\begin{aligned}
\sum_b \mathcal{N}_{\lambda a}{}^b\, \hat{q}^\omega_b &= \sum_{\gamma'} b^\lambda{}_{\gamma'} \sum_{b'\in P'_+} N'_{\gamma' a'}{}^{b'}\, \dim_B(b') \pmod{M_B} \\
&= \sum_{\gamma'} b^\lambda{}_{\gamma'}\, \dim_B(\gamma')\, \dim_B(a') \pmod{M_B} \\
&= \dim_A(\lambda)\, \dim_B(a') \pmod{M_B}\,. \qquad (3.14)
\end{aligned}
$$

Hence the ansatz (3.10) solves (1.7) with $M^\omega = M_B$, where $M = M_B$ is the untwisted charge number corresponding to $\widehat{B}_n$ at level $k+2$. Using conjecture B [2] this equals

$$
M_B = \frac{k+2n+1}{\gcd\{k+2n+1, L_B\}}\,, \qquad L_B = \mathrm{lcm}\{1,2,\ldots,2n-1\}\,. \qquad (3.15)
$$

Since k is even, $k+2n+1$ is odd, and thus M_B agrees, for $n \geq 2$, with the expression for $\mathrm{su}(2n+1)$ at level k. It follows that the ansatz (3.10) solves (1.7) with $M^\omega = M$.

Finally, (3.10) is equivalent to our proposed ansatz (3.1). Indeed, as is explained in appendix A, we have the relation

$$
2^n \dim_C(a) = \dim_B(a')\,. \qquad (3.16)
$$

Thus the charges (3.10) differ by an overall factor $l = 2^n$ from (3.1). Since k is even, M^ω is odd, and thus 2^n is coprime to (and invertible modulo) M^ω. Hence (3.10) is equivalent to (3.1); in particular, (3.1) therefore also satisfies (1.7) with $M^\omega = M$.

4. The analysis for $\mathrm{su}(2n)$

Next we turn to $\mathrm{su}(2n) = A_{2n-1}$ with $\omega = C$, for which the twisted algebra behaves differently from $\mathrm{su}(2n+1)$. As was explained in [5, 7], the ω-twisted boundary states are labelled by the level k ω-twisted weights of $\mathfrak{g} = \widehat{\mathrm{su}}(2n)$, or alternatively the level k weights of the twisted Lie algebra $\mathfrak{g}^\omega = A^{(2)}_{2n-1}$. They can be equated with all $(n+1)$-tuples $(a_0; a_1, \ldots, a_n)$ where $k = a_0 + a_1 + 2a_2 + \cdots + 2a_n$ and $a_i \in \mathbb{Z}_{\geq 0}$. Once again, the invariant subalgebra $\bar{\mathfrak{g}}^\omega$ is C_n, and the twisted D-brane charge q^ω_a is given by (3.1).

In [7] we remarked that the ψ-matrix can be interpreted as a submatrix of S (rescaled by $\sqrt{2}$) for $\widehat{B}_n$ level $k+1$, using the identifications $\mu \mapsto \mu'' = (\mu_0+\mu_1+1; \mu_1, \ldots, \mu_n) \in P'_+$ (for ω-invariant weights $\mu \in \mathcal{E}$) and $a \mapsto a' = (a_0; a_1, \ldots, a_{n-1}, 2a_n+1) \in P'_+$. [The ω-invariant weights $\mu \in \mathcal{E}$ are characterised by the condition $k \geq 2\mu_1 + \cdots + 2\mu_{n-1} + \mu_n$.] As before we use primes to denote $\widehat{B}_n$ level $k+1$ quantities. In order to find an expression for the NIM-rep coefficients in terms of $\widehat{B}_n$ fusions at $k' = k+1$, we need to express the $\widehat{A}_{2n-1}$ ratios $S_{\lambda\mu}/S_{0\mu}$ in terms of the $\widehat{B}_n$ ratios $S'_{\gamma\mu''}/S'_{0\mu''}$. The usual way to do this involves branching rules, but there is no embedding of $\mathrm{so}(2n+1) = B_n$ into $\mathrm{su}(2n)$. Fortunately, it is possible, nonetheless, to write any $\mathrm{su}(2n)$ character, restricted to ω-invariant weights $\mu \in \mathcal{E}$, in terms of $\mathrm{so}(2n+1)$ characters (this possibility was missed in [7]). This is done by comparing the embeddings $\mathrm{sp}(2n) \subset \mathrm{su}(2n)$ and $\mathrm{sp}(2n) \subset \mathrm{so}(2n+1)$. What we find is that, for any $1 \le m < n$ and any μ,

$$\mathrm{ch}_{\Lambda_m}[\mu] = \mathrm{ch}_{\Lambda_{2n+1-m}}[\mu] = \sum_{i=0}^{m} (-1)^i \, \mathrm{ch}'_{\Lambda'_{m-i}}[\mu''] \,, \tag{4.1}$$

and

$$\mathrm{ch}_{\Lambda_n}[\mu] = \mathrm{ch}'_{2\Lambda'_n}[\mu''] + \sum_{i=0}^{n-1} (-1)^i \, \mathrm{ch}'_{\Lambda'_{n-i}}[\mu''] \,, \tag{4.2}$$

where we have defined

$$\mathrm{ch}_\lambda[\mu] \equiv \mathrm{ch}_\lambda \left(-2\pi i(\mu+\rho)/(k+2n)\right) = \frac{S_{\lambda\mu}}{S_{0\mu}} \,,$$

and similarly for $\mathrm{ch}'_{\lambda'}[\mu'']$. Here, the Λ_i and Λ'_i are the fundamental weights for A_{2n-1} and B_n, respectively. More generally, we can write

$$\mathrm{ch}_\lambda[\mu] = \sum_\gamma b^\lambda{}_\gamma \, \mathrm{ch}'_\gamma[\mu''] \,,$$

where these branching rule-like coefficients $b^\lambda{}_\gamma$ are necessarily integers but can be negative.[6] Note that all γ appearing in these decompositions will be non-spinors (since this is true of the fundamental weights Λ_m).

We can now compute

$$\mathcal{N}_{\lambda a}{}^b = \sum_{\mu \in \mathcal{E}} \psi_{a\mu} \frac{S_{\lambda\mu}}{S_{0\mu}} \psi_{b\mu} = 2 \sum_{\mu'' \in \mathcal{G}'} \sum_\gamma S'_{a'\mu''} \, b^\lambda{}_\gamma \, \frac{S'_{\gamma\mu''}}{S'_{0\mu''}} \, S'_{b'\mu''} \,,$$

where the set $\mathcal{G}'$ consists of all images $\mu \mapsto \mu'' \in P_+^{k+1}(B_n)$, i.e. of all $\mu'' \in P_+^{k+1}(B_n)$ for which $\mu''_0 > \mu''_1$. We need to extend the sum over μ'' to all of $P_+^{k+1}(B_n)$. For this, we observe that since $\mathcal{G}'$ consists of all μ'' with $\mu''_0 > \mu''_1$, any weight $\nu \in P_+^{k+1}(B_n)$ is either exclusively in $\mathcal{G}'$, or $J'\nu$ is, or $\nu = J'\nu$ is a fixed point. Because a' is a spinor, if $\nu = J'\nu$ then $S'_{a'\nu} = S'_{a',J'\nu} = -S'_{a'\nu}$ and so $S'_{a'\nu} = 0$. Because a' and b' are spinors and γ is not, $S'_{a',J'\nu} S'_{\gamma,J'\nu} S'_{b',J'\nu} / S'_{0,J'\nu} = S'_{a'\nu} S'_{\gamma\nu} S'_{b'\nu} / S'_{0\nu}$. Thus we obtain

$$\mathcal{N}_{\lambda a}{}^b = \sum_\gamma b^\lambda{}_\gamma N'_{\gamma a'}{}^{b'} \,, \tag{4.3}$$

where $N'_{a'\gamma}{}^{b'}$ is the $\widehat{B}_n$ fusion rule at $k' = k+1$. In particular, we obtain from (4.1) the nice formula

$$\mathcal{N}_{\Lambda_m a}{}^b = \sum_{i=0}^{m} (-1)^i \, N'_{\Lambda'_{m-i} a'}{}^{b'} \,, \tag{4.4}$$

for any $1 \le m < n$, with the obvious minor modification for $m = n$. Incidentally, the NIM-rep for the simple current J^i is trivial for i even, and for i odd corresponds to the interchange $a_0 \leftrightarrow a_1$.

[6] Virtual branching rules of this kind were considered before in [26] where they were called 'subjoinings'; we thank Mark Walton for bringing this reference to our attention.

As in the previous case we now make the ansatz (see (3.10)) $\hat{q}^{\omega}_a = \dim_B(a')$. In order to verify that this obeys (1.7), we observe

$$\begin{aligned}
\dim_A(\lambda)\dim_B(a') &= \sum_\gamma b^\lambda{}_\gamma \dim_B(\gamma)\dim_B(a') \\
&= \sum_\gamma \sum_b b^\lambda{}_\gamma N'_{\gamma a'}{}^{b'} \dim_B(b') \\
&= \sum_b \mathcal{N}_{\lambda a}{}^b \dim_B(b') \pmod{2^n M_B}\,,
\end{aligned} \tag{4.5}$$

using Conjecture $\mathrm{B}^{\mathrm{spin}}$. If conjecture B holds, then M_B equals manifestly M_A (compare (1.6) with (1.5)). Also, we know $\dim_B(a') = 2^n \dim_C(a)$, from appendix B. Thus we can divide (4.5) by 2^n and the desired result follows.

As is explained in appendix D, we can *prove* Conjecture $\mathrm{B}^{\mathrm{spin}}$ (assuming Conjecture B) provided 4 does not divide M_B. For example this is automatic provided k and $-2n$ are not congruent mod 8. In particular, this covers therefore the case when k is odd.

Finally, let us sketch why the NIM-rep coefficient $\mathcal{N}_{\lambda a}{}^b$ does indeed equal the corresponding tensor product coefficient $T_{\lambda a}{}^b$ for the invariant subalgebra $\overline{\mathfrak{g}}^\omega = C_n$, for sufficiently large level. One way to see this uses the expression, derived in [19], for $\mathcal{N}$ in terms of fusions $\widehat{N}$ for $\mathrm{sp}(2n) = C_n$ at level $k+n-1$:[7]

$$\mathcal{N}_{\lambda a}{}^b = \sum_{l=0}^{n-1} \sum_{n\geq i_1 > \cdots > i_l > 1} (-1)^{\lfloor \frac{i_1}{2}\rfloor + \cdots + \lfloor \frac{i_l}{2}\rfloor} \sum_c b^\lambda{}_c \widehat{N}_{ac}{}^{\sigma_{i_1}\cdots\sigma_{i_l}\sigma_1^{i_1+\cdots+i_l}(b)}\,, \tag{4.6}$$

where the $b^\lambda{}_c$ are the $\mathrm{sp}(2n) \subset \mathrm{su}(2n)$ branching coefficients, and where

$$\sigma_i[a_1,\dots,a_n] = [a_{i-1},\dots,a_1, k+2n - \sum_{j=1}^{i} a_j - 2\sum_{j=i+1}^{n} a_j, a_{i+1},\dots,a_n]\,.$$

For k much larger than $\sum_{i=1}^n(\lambda_i + a_i)$, it is easy to see that for any choice of $n \geq i_1 > \cdots > i_l > 1$ (for $l > 0$) some component of $\sigma_{i_1}\cdots\sigma_{i_l}\sigma_1^{i_1+\cdots+i_l}(b)$ will be large — e.g., if $i_1 + \cdots + i_1$ is odd it will be the $(i_1 - i_2 + \cdots \pm i_l \mp 1)$-th one. For this reason, all fusion coefficients $\widehat{N}$ in (4.6) will vanish except the $l = 0$ one, and we obtain $\mathcal{N}_{\lambda a}{}^b = \sum_c b^\lambda{}_c \widehat{N}_{ac}{}^b = T_{\lambda a}{}^b$.

5. The analysis for $\mathrm{so}(2n)$

The final algebras we will discuss are $\mathrm{so}(2n) = D_n$ with the automorphism ω corresponding to chirality-flip (i.e. ω interchanges the Dynkin labels $\lambda_{n-1} \leftrightarrow \lambda_n$). The boundary states are all n-tuples $(a_0; a_1, \dots, a_{n-1})$ for which $k = a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1}$. Let primes denote the $\widehat{B}_{n-1}$ level $k+1$ quantities as usual. The NIM-rep coefficients can be interpreted [7] in terms of fusions N', and branching coefficients $b^\lambda{}_{\gamma'}$ of $B_{n-1} \subset D_n$,

$$\mathcal{N}_{\lambda a}{}^b = \sum_{\gamma'} b^\lambda{}_{\gamma'} \left(N'_{\gamma' a'}{}^{b'} - N'_{\gamma' a'}{}^{J'b'}\right)\,, \tag{5.1}$$

[7] The fact that a given NIM-rep may be related to different fusion rules was first observed in [27].

where $a' = (a_0 + a_1 + 1; a_1, \ldots, a_{n-1}) \in P'_+ \equiv P_+^{k+1}(B_{n-1})$. In particular,

$$\mathcal{N}_{\Lambda_i a}{}^b = N'_{\Lambda'_i a'}{}^{b'} + N'_{\Lambda'_{i-1} a'}{}^{b'}, \tag{5.2}$$

for $i < n-1$ (where the term $N'_{\Lambda'_0 a'}{}^{b'}$ is defined to be $\delta_a{}^b$), as well as

$$\mathcal{N}_{\Lambda_n a}{}^b = \mathcal{N}_{\Lambda_{n-1} a}{}^b = N'_{\Lambda'_{n-1} a'}{}^{b'}. \tag{5.3}$$

The simple current matrix $\mathcal{N}_{J_v}$ is trivial, while the simple current matrices $\mathcal{N}_{J_s}$ and $\mathcal{N}_{J_c}$ are equal and correspond to the symmetry of the $D_n^{(2)}$ Dynkin diagram, i.e. they send a to $(a_n; a_{n-1}, \ldots, a_0)$.

The invariant subalgebra $\bar{\mathfrak{g}}^\omega$ here is B_{n-1}. Our ansatz for the charge is

$$q_a^\omega = \dim_B([a_1, \ldots, a_{n-1}]) = \dim_B(a'). \tag{5.4}$$

The right hand side of (1.7) is then

$$\sum_b \mathcal{N}_{\lambda a}{}^b q_b^\omega = \sum_{\gamma'} b^\lambda{}_{\gamma'} \Big[\sum_{b' \in \mathcal{G}'} N'_{\gamma' a'}{}^{b'} \dim_B(b') - \sum_{b' \in \mathcal{G}'} N'_{\gamma' a'}{}^{J'b'} \dim_B(b') \Big], \tag{5.5}$$

where $\mathcal{G}'$ is the subset of P'_+ that appeared already in section 4. As explained there, any $b \in P'_+$ is either in $\mathcal{G}'$, or $J'b$ is in $\mathcal{G}'$, or $J'b = b$. Using as before the fact that $\dim_B(J'b) = -\dim_B(b)$ modulo M_B, we therefore have that

$$\begin{aligned} \sum_b \mathcal{N}_{\lambda a}{}^b q_b^\omega &= \sum_{\gamma'} b^\lambda{}_{\gamma'} \Big[\sum_{b \in P'_+} N'_{\gamma' a'}{}^b \dim_B(b) - \sum_{J'b = b} N'_{\gamma' a'}{}^b \dim_B(b) \Big] \pmod{M_B} \\ &= \dim_D(\lambda)\, q_a^\omega - \sum_{\gamma'} b^\lambda{}_{\gamma'} \sum_{J'b=b} N'_{\gamma' a'}{}^b \dim_B(b) \pmod{M_B}. \end{aligned} \tag{5.6}$$

Using conjecture B and D, the explicit formulae (1.6) imply that M_B for $\widehat{B}_{n-1}$ at level $k+1$ equals M_D for $\widehat{D}_n$ at level k, and thus the identity holds modulo M_D. Thus if we can ignore the last term, the ansatz (5.4) solves (1.7) with $M^\omega = M_D$.

It therefore only remains to analyse the last term in (5.6). Since $\dim_B(J'b) = -\dim_B(b)$ modulo M_B, if b is a fixed point of J' we have $2\dim_B(b) = 0$ modulo M_B. If M_B is odd, as is the case for example when k is odd, then $\dim_B(b) = 0$ modulo M_B, and we can ignore the last term in (5.6). This leaves us with the case where both k and M_B are even. It is easy to see that $J'b = b$ implies that $b_{n-1} = k+1 \pmod 2$. Thus for k even the fixed point is necessarily a spinor of B_{n-1}, so 2^n must divide its dimension (see the argument given in detail in the next section). Provided 2^{n+1} does not divide M_B, the facts that $M_B/2$ and 2^n both divide $\dim_B(b)$ implies that M_B itself must divide $\dim_B(b)$, and thus we can again ignore the last term in (5.6).

This leaves with the case when 2^{n+1} does divide M_B. In this case we use Conjecture $\mathrm{B}^{\mathrm{spin}}$ to deduce that $2^n M_B$ divides $(\dim(J'0) - 1)\dim_B(b)$. Together with the fact (proved in appendix B) that $\dim_B(J'0) = -1 + \ell M_B$ for some $\ell \in \mathbb{Z}$, we can actually deduce the stronger result that $2^{n-1} M_B$ divides $\dim_B(b)$. Thus we can again ignore the last term in (5.6), and we are done.

It is clear that, for large k, the NIM-rep coefficient $\mathcal{N}_{\lambda a}{}^b$ in (5.1) becomes the $\bar{\mathfrak{g}}^\omega = B_{n-1}$ tensor product coefficient $T'_{\lambda a}{}^b$: for large k, the term $N'_{\gamma' a'}{}^{J'b'}$ vanishes.

6. Uniqueness

In this section we prove statement (ii) of section 1.2. Consider first the case of su$(2n)$. Choose an integer $q'_{b'}$ for each spinor $b' \in P_+^{k+1}(B_n)$, and an integer m' such that

$$\dim_A(\lambda)\, q'_{b'} 2^n = \sum_{\gamma,c'} b^\lambda{}_\gamma N'_{b'\gamma}{}^{c'}\, q'_{c'} 2^n \pmod{2^n m'}\,, \tag{6.1}$$

for all $\lambda \in P_+^k(A_n)$ and all spinors $b' \in P_+^{k+1}(B_n)$ (the multiplication by 2^n is for later convenience). We require $\gcd q'_{b'} = 1$. Then $2^n m'$ divides $2^n M_A$ (by the argument in section 1.2), and so must also divide $m_B^{\rm spin}$ (by Conjecture $B^{\rm spin}$).

It is a classical result (see e.g. [28]) that any B_n character ${\rm ch}'_{b'}$ can be expressed as a polynomial over the integers $\mathbb{Z}$ in the fundamental characters ${\rm ch}'_{\Lambda'_i}$ $(i \le n)$. If the weight b' is a spinor, then every term in this polynomial will contain an odd number of ${\rm ch}'_{\Lambda'_n}$'s. But

$$({\rm ch}'_{\Lambda'_n})^2 = 1 + \sum_{i=1}^{n-1} {\rm ch}'_{\Lambda'_i} + {\rm ch}'_{2\Lambda'_n}\,, \tag{6.2}$$

and so the character ${\rm ch}'_{b'}$ of any B_n spinor can be expressed as the product of ${\rm ch}'_{\Lambda'_n}$ with some polynomial in ${\rm ch}'_{\Lambda'_i}$ $(i < n)$ and ${\rm ch}'_{2\Lambda'_n}$, or more precisely some combination ${\rm ch}'_{b'-\Lambda'_n} + \sum_j \ell'_j {\rm ch}'_{\gamma^{(j)}}$ of non-spinors (for $\ell'_j \in \mathbb{Z}$). By construction, each $\gamma^{(j)} + \Lambda'_n \prec b'$, i.e. $b' - \gamma^{(j)} - \Lambda'_n$ is a nonzero sum of positive roots. Now, from (4.1), (4.2) we obtain ${\rm ch}'_{\Lambda'_i} = {\rm ch}_{\Lambda_i} - {\rm ch}_{\Lambda_{i-1}}$, for $i < n$, and ${\rm ch}'_{2\Lambda'_n} = {\rm ch}_{\Lambda_n} - {\rm ch}_{\Lambda_{n-1}}$. Thus the character ${\rm ch}'_{b'}$ of any B_n spinor can be expressed as the product of ${\rm ch}'_{\Lambda'_n}$ with some combination $\sum_i \ell_i {\rm ch}_{\lambda^{(i)}}$ (with integer coefficients ℓ_i) of characters of A_n weights $\lambda^{(i)}$.

We are interested here in Weyl dimensions, i.e. in the evaluation of these characters at 0. In particular we obtain

$$\begin{aligned} \dim_B(b') q'_{\Lambda'_n} &= \sum_i \ell_i \dim_A(\lambda^{(i)})\, 2^n\, q'_{\Lambda'_n} \\ &= \sum_{c'} (N'_{b'-\Lambda'_n,\Lambda'_n}{}^{c'} + \sum_j \ell'_j\, N'_{\gamma^{(j)}\Lambda'_n}{}^{c'}) q'_{c'} 2^n \pmod{2^n m'}\,, \end{aligned} \tag{6.3}$$

where $c' \prec b'$ are spinors. Inductively, we get $q'_{b'} 2^n = \dim_B(b') q'_{\Lambda'_n} \pmod{2^n m'}$. Hence $1 = \gcd q'_{b'} = \gcd\{q'_{\Lambda'_n}, m'\}$, and we get statement (ii) of section 1.2.

Thus there is a unique solution to (1.7) for su$(2n)$, up to equivalence. Note that uniqueness here follows for much the same reason uniqueness occurs for the untwisted case: there is a boundary condition here (namely $a = [0, \ldots, 0]$) which acts as an identity. Uniqueness for su$(2n+1)$ at odd level follows immediately from the $\widehat{C}_n$ level $(k-1)/2$ analysis of [2]. The uniqueness argument for su$(2n+1)$ when k is even is similar to that given above for su$(2n)$: by those arguments we get $q'_a = \dim_B(a') = -\dim_B(J'a') \pmod{m'}$ for boundary states $a \in \mathcal{B}$, and uniqueness follows. The argument for so$(2n)$ is the same as that of su$(2n+1)$ at even level.

7. Conclusion

In this paper we have shown (under the assumptions detailed in 1.2) that the charge group of the ω-twisted D-branes of WZW models agrees with that of the untwisted D-branes. This is in nice agreement with the recent K-theory calculation [14, 3].

We have also shown that the charge of the ω-twisted D-brane corresponding to the twisted representation a has a simple interpretation: it is the dimension of the conformal highest weight space of the representation a. As for the case of an untwisted D-brane, the charge associated to the D-brane has therefore a simple string theoretic interpretation: it is the multiplicity of the ground state in the open string between the fundamental D0-brane (the untwisted D-brane corresponding to the identity representation) and the brane in question. [This follows simply from the fact that the states of the open string between the fundamental D0-brane and any D-brane labelled by x (where x is either an untwisted representation μ or a twisted representation a) transforms in the irreducible representation x.] In the full supersymmetric formulation of the WZW models, this number therefore has an interpretation as an intersection number. This observation may help to give a more conceptual argument for why the twisted and the untwisted charge groups should coincide.

In this paper we have restricted attention to the case of the classical algebras with ω of order two. We expect that the remaining cases (namely E_6 with ω being charge conjugation, and SO(8) with ω being triality) should work out similarly. Our arguments relied on a number of intriguing Lie theoretic properties. It would be nice to find more conceptual arguments for them.

We have also found simple formulae for the NIM-rep coefficients for the fundamental representations. These NIM-rep coefficients are truncations of the tensor product coefficients of the invariant subalgebra. This gives strong support to the suggestion that these NIM-reps are in fact the twisted fusion rules.

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A. The proof of the miraculous dimension formula

The purpose of this subsection is to prove, for any $n \geq 2$ and any highest weight $\lambda = [\lambda_1, \dots, \lambda_n]$ of $\mathrm{sp}(2n) = C_n$, the identity

$$2^n \dim_C(\lambda) = \dim_B(\lambda'), \tag{A.1}$$

relating the Weyl dimension of the C_n weight λ to that of the B_n weight $\lambda' = [\lambda_1, \dots, \lambda_{n-1}, 2\lambda_n + 1]$. For notational clarity, in this appendix we shall let primes denote B_n

quantities (or sometimes $\widehat{B}_n$ level k quantities), while C_n (or $\widehat{C}_n$ level $k+n-2$) quantities are unprimed.

The denominator identity of the simple Lie algebras yields a useful expression for the quantum dimensions. For $\widehat{B}_n$ level k, let $\kappa = k+2n-1$ and write $\mu[i]' = \sum_{j=i}^{n-1}(\mu_j+1) + (\mu_n+1)/2$ as usual. Then

$$\frac{S'_{\mu 0}}{S'_{00}} = \prod_{1\le i<j\le n} \frac{\sin(\pi(\mu[i]'-\mu[j]')/\kappa)}{\sin(\pi(0[i]'-0[j]')/\kappa)} \frac{\sin(\pi(\mu[i]'+\mu[j]')/\kappa)}{\sin(\pi(0[i]'+0[j]')/\kappa)} \prod_{i=1}^{n} \frac{\sin(\pi\,\mu[i]'/\kappa)}{\sin(\pi\,0[i]'/\kappa)}\,. \tag{A.2}$$

For $\widehat{C}_n$ level $k+n-2$, let $\mu[i] = \sum_{j=i}^{n}(\mu_j+1)$ as usual. Then

$$\frac{S_{\mu 0}}{S_{00}} = \prod_{1\le i<j\le n} \frac{\sin(\pi(\mu[i]-\mu[j])/(2\kappa))}{\sin(\pi(0[i]-0[j])/(2\kappa))} \frac{\sin(\pi(\mu[i]+\mu[j])/(2\kappa))}{\sin(\pi(0[i]+0[j])/(2\kappa))} \prod_{i=1}^{n} \frac{\sin(\pi\,\mu[i]/\kappa)}{\sin(\pi\,0[i]/\kappa)}\,. \tag{A.3}$$

The Weyl dimension formula is obtained from these by taking the $k\to\infty$ limit, and using the asymptotic formula $\sin(x)\approx x$ for x small.

Note that each $\lambda'[i]' = \lambda[i]$. Thus, the ratio of the B_n quantum dimension of λ' with the B_n quantum dimension of the spinor $0' = \Lambda_n$, is given by the formula (A.3) (for the same value of κ), except that the '2κ's there are replaced with 'κ's. Taking the $k\to\infty$ limit, and noting that the dimension of the B_n spinor Λ_n is 2^n, we get the desired (A.1).

It would be interesting to obtain a more conceptual, algebraic proof of (A.1), as opposed to our direct calculation from the Weyl dimension formula. Two observations along these lines can be made. The first is that the source of the 2^n factor is presumably the B_n fundamental spinor. The second is that this relation (A.1) clearly does not generalise naturally to quantum dimensions, and thus cannot be obtained merely by playing with tensor products or character manipulations. It is for making this second point that we wrote down the preceding *quantum* dimension formulae, rather than the simpler *Weyl* dimension formulae.

B. Dimensions of simple currents

Let J be the simple current of $\widehat{B}_n$ level k'. So $J[a_1,\dots,a_n] = [k'-a_1-2a_2-\cdots-2a_{n-1}-a_n, a_2, a_3, \dots, a_n]$. Put $\kappa = k'+2n-1$, and $M_B = \kappa/\gcd\{L_B,\kappa\}$ as usual, where L_B is as in (1.6).

Claim. Let ν be any weight of B_n level k'. Then the dimensions of $J\nu$ and ν are related by

$$\dim_B(J\nu) = -\dim_B(\nu) \pmod{M_B}\,. \tag{B.1}$$

To prove this, first note that it is sufficient to show that the dimension $\dim_B(J0) = \dim_B[k',0,\dots,0]$ is congruent to -1 (mod M_B). From (A.2) we obtain

$$\dim_B(J0) = \binom{\kappa}{2n-1} + \binom{\kappa-1}{2n-1}\,. \tag{B.2}$$

Let p be any prime dividing M_B. Let p^α and p^β be the exact powers of p dividing κ and L_B, respectively (we say p^a *exactly divides* N if N/p^a is coprime to p). So we know that the largest power of p which divides any number $1 \leq \ell \leq 2n-1$, is p^β. We also know $\alpha > \beta \geq 0$ and $p^{\alpha-\beta}$ exactly divides M_B.

Prime factorisations apply to rational numbers r as for integers. Once again we get a unique factorisation $r = \pm \prod_i p_i^{a_i}$, where now the exponents $\exp_{p_i}(r) \equiv a_i \in \mathbb{Z}$ can be negative. We say that p^a *divides* r if $a \leq \exp_p(r)$, and we say that p is *coprime to* r, if $\exp_p(r) = 0$. For example, $\exp_2(.75) = -2$, and 5 is coprime to .75.

We will show that $p^{\alpha-\beta}$ divides

$$\binom{\kappa}{2n-1} = \frac{\kappa}{2n-1} \prod_{\ell=1}^{2n-2} \frac{\kappa-\ell}{\ell},$$

and that

$$\binom{\kappa-1}{2n-1} = \prod_{\ell=1}^{2n-1} \frac{\kappa-\ell}{\ell}$$

is congruent to -1 (mod M_B). To see this, take any $1 \leq \ell \leq 2n-1$ and let $i = \exp_p(\ell)$. Then $i \leq \beta < \alpha$ and p is coprime to the rational number $(\kappa-\ell)/\ell = ((\kappa/p^i)-(\ell/p^i))/(\ell/p^i)$. This immediately gives us the first statement, because $p^{\alpha-\beta}$ clearly divides the first fraction. To get the second statement, note that ℓ/p^i is coprime to p and hence is invertible mod $p^{\alpha-\beta}$, and so we obtain

$$\frac{\kappa-\ell}{\ell} = \left(\left(\frac{\kappa}{p^i}\right) - \left(\frac{\ell}{p^i}\right)\right)\left(\frac{\ell}{p^i}\right)^{-1} = \left(0 - \left(\frac{\ell}{p^i}\right)\right)\left(\frac{\ell}{p^i}\right)^{-1} = -1 \quad (\mathrm{mod}\ p^{\alpha-\beta})\,.$$

C. The example of su(7)

In this appendix we describe the NIM-rep for su(7), level $k = 1, 2, 3$. Let us recall that the boundary states are labelled by triples of integers $a = [a_1, a_2, a_3]$, where $k \geq 2a_1 + 2a_2 + 2a_3$. These triples can be identified with a subset of the C_3 weights at $k_C = k+3$. To each such weight, associate the B_3 weight $a' = [a_1, a_2, 2a_3+1]$; this defines a dominant highest weight for B_3 at level $k_B = k+2$. Let us denote the first fundamental representation of su(7) by $\mathbf{7}$, and likewise for so(7). Then our NIM-rep formula is simply

$$\mathcal{N}_{\mathbf{7}a}{}^{b} = N'_{\mathbf{7}a'}{}^{b'} - N'_{\mathbf{7}a'}{}^{J'b'}\,, \tag{C.1}$$

where $N'_{\mathbf{7}a'}{}^{b'}$ are the fusion rules of so(7) at level $k_B = k+2$. In this manner one easily calculates

$$\begin{aligned}
\mathbf{7} \otimes [0,0,0]_1 &= [0,0,0]_1 \oplus [1,0,0]_2 \\
\mathbf{7} \otimes [1,0,0]_2 &= [0,0,0]_2 \oplus [1,0,0]_2 \oplus [0,1,0]_2 \oplus [2,0,0]_4 \\
\mathbf{7} \otimes [0,1,0]_2 &= [1,0,0]_2 \oplus [0,0,1]_2 \oplus [0,1,0]_3 \oplus [1,1,0]_4 \\
\mathbf{7} \otimes [0,0,1]_2 &= [0,1,0]_2 \oplus [0,0,1]_3 \oplus [1,0,1]_4\,,
\end{aligned} \tag{C.2}$$

where $[a_1, a_2, a_2]$ label the boundary states (in C_3 notation), and the index denotes the value of the level k for which the relevant representation appears first in the NIM-rep. These twisted fusion rules are indeed a truncation of the C_3 tensor product coefficients: under the embedding of C_3 into su(7), the **7** of su(7) becomes $[0,0,0]\oplus[1,0,0]$. Equation (C.2) should therefore be compared with

$$\begin{aligned}
\Big([0,0,0]\oplus[1,0,0]\Big)\otimes[0,0,0] &= [0,0,0]\oplus[1,0,0]\\
\Big([0,0,0]\oplus[1,0,0]\Big)\otimes[1,0,0] &= [0,0,0]\oplus[1,0,0]\oplus[0,1,0]\oplus[2,0,0]\\
\Big([0,0,0]\oplus[1,0,0]\Big)\otimes[0,1,0] &= [1,0,0]\oplus[0,0,1]\oplus[0,1,0]\oplus[1,1,0]\\
\Big([0,0,0]\oplus[1,0,0]\Big)\otimes[0,0,1] &= [0,1,0]\oplus[0,0,1]\oplus[1,0,1]\,.
\end{aligned}\tag{C.3}$$

For large k, these two expressions agree indeed.

D. Verification of Conjecture B$^{\text{spin}}$ for most levels

We know [29] that the fusion ring of $\widehat{B}_n$ level k', for example, is the quotient of the character ring of B_n, say, by the fusion ideal $\mathcal{I}_{k'}$, which can be thought of as all linear combinations (with coefficients in $\mathbb{Z}$) in the B_n characters ch_λ, which vanish at all values $-2\pi i(\mu+\rho)/(k'+2n-1)$ for $\mu\in P_+^{k'}(B_n)$. We can split $\mathcal{I}_{k'}$ into a sum $\mathcal{I}_n+\mathcal{I}_s$ of spinor and non-spinor contributions. By $\dim_B(\sum\ell_i\mathrm{ch}_{b^{(i)}})$ we mean $\sum\ell_i\dim_B(b^{(i)})$ (or equivalently, the specialisation of that character sum to 0). Let M_n (resp. M_s) be the gcd of all $\dim_B(\sum\ell_i\mathrm{ch}_{b^{(i)}})$, as one runs over all elements of $\mathcal{I}_n$ (resp. $\mathcal{I}_s$). Then $m_B=\gcd\{M_n,M_s\}$ and $m_B^{\text{spin}}=M_s$.

Now, $\mathcal{I}_{k'}$ is an ideal, so $\mathrm{ch}_{\Lambda_n}\mathcal{I}_{k'}$ is contained in $\mathcal{I}$, and hence $\mathrm{ch}_{\Lambda_n}\mathcal{I}_n\subset\mathcal{I}_s$ and $\mathrm{ch}_{\Lambda_n}\mathcal{I}_s\subset\mathcal{I}_n$. Thus M_s divides 2^nM_n and M_n divides 2^nM_s. In particular, m_B,M_s, and M_n differ from one another only by powers of 2.

As we know (see section 6), 2^n must divide the dimension of any spinor, and hence 2^n must divide M_s. Thus as long as 2^n does not divide m_B (e.g. if m_B is odd), m_B must equal M_n.

Consider first m_B odd. Then we know 2^n divides M_s and M_s divides 2^nM_n. Thus $M_s=2^nM_n$, in agreement with conjecture B$^{\text{spin}}$.

Now suppose m_B is even (so the level k' is necessarily odd). From the reasoning of section 6, we find that every element of $\mathcal{I}_s$ is of the form $\mathrm{ch}_{\Lambda_n}\sum\ell_j\mathrm{ch}_{b^{(j)}}$ where each $b^{(j)}$ is a non-spinor. It can be shown that, for $\mu\in P_+^{k'}(B_n)$, $\mathrm{ch}_{\Lambda_n}[\mu]=0$ iff μ is a fixed point of J. (Proof: If $J\mu=\mu$, then $S_{\Lambda_n\mu}=S_{\Lambda_n,J\mu}=-S_{\Lambda_n\mu}$. So $S_{\Lambda_n\mu}=0$. Conversely, if $S_{\Lambda_n\mu}=0$, then

$$\frac{S_{\Lambda_i\mu}}{S_{0\mu}}=\frac{S_{\Lambda_i,J\mu}}{S_{0,J\mu}}\qquad\forall i=1,\dots,n\,,$$

and so $\mu=J\mu$.)

Thus $\mathrm{ch}_{\Lambda_n}\sum\ell_j\mathrm{ch}_{b^{(j)}}\in\mathcal{I}_s$ iff $\sum\ell_j\mathrm{ch}_{b^{(j)}}[\mu]=0$ for all non-fixed points $\mu\in P_+^{k'}(B_n)$. In particular, this means $\sum\ell_j\mathrm{ch}_{b^{(j)}}[\mu]=0$ for all non-spinors $\mu\in P_+^{k'}$. Thus $\sum\ell_j\mathrm{ch}_{b^{(j)}}[\mu]=-\sum\ell_j\mathrm{ch}_{Jb^{(j)}}[\mu]$ for all $\mu\in P_+^{k'}$. This means that $\sum\ell_j\mathrm{ch}_{b^{(j)}}$ is a sum of terms of the form $\mathrm{ch}_b-\mathrm{ch}_{Jb}$. Since $\dim_B(\mathrm{ch}_\nu-\mathrm{ch}_{J\nu})=2\nu\pmod{m_B}$ and m_B is even, we get that $\dim_B(\sum\ell_j\mathrm{ch}_{b^{(j)}})$ must also be even.

Thus 2^{n+1} must divide M_s. If 4 does not divide m_B, then M_s dividing $2^n m_B$ requires $M_s = 2^n m_B$, in agreement with conjecture $\mathrm{B}^{\mathrm{spin}}$.

Note that, since L_B is always even, 4 can divide M_B only if 8 divides $k' + 2n - 1$.

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